

The Gilbert Arborescence Problem*

M. G. Volz[†] M. Brazil[†] C. J. Ras[†] K. J. Swanepoel[‡]
D. A. Thomas[†]

September 23, 2009

Abstract—We investigate the problem of designing a minimum cost flow network interconnecting n sources and a single sink, each with known locations and flows. The network may contain other unprescribed nodes, known as Steiner points. For concave increasing cost functions, a minimum cost network of this sort has a tree topology, and hence can be called a Minimum Gilbert Arborescence (MGA). We characterise the local topological structure of Steiner points in MGAs, showing, in particular, that for a wide range of metrics and cost-functions the degree of each Steiner point is 3.

Keywords: *Gilbert network, minimum cost networks, network flows, Steiner trees*

1 Introduction

The *Steiner Minimum Tree (SMT) problem* asks for a shortest network spanning a given set of nodes (*terminals*) in a given metric space. It differs from the minimum spanning tree problem in that additional nodes, referred to as *Steiner points*, can be included to create a spanning network that is shorter than would otherwise be possible. This is a fundamental problem in physical network design optimisation, and has numerous applications, including the design of telecommunications or transport networks for the problem in the Euclidean plane (the l_2 metric), and the design of microchips for the problem in the rectilinear plane (the l_1 metric) [5].

Gilbert [4] proposed a generalisation of the SMT problem whereby symmetric non-negative *flows* are assigned between each pair of terminals. The aim is to find a least cost network interconnecting the terminals, where each edge has an associated total flow such that the flow conditions between terminals are satisfied, and Steiner points satisfy Kirchhoff's rule (ie, the net incoming and outgoing flows at each Steiner point are equal). The cost of

*This research was supported by an ARC Linkage Grant with Newmont Australia Limited.

[†]ARC Special Research Centre for Ultra-Broadband Information Networks, Department of Electrical and Electronic Engineering, The University of Melbourne, Victoria 3010, Australia. CUBIN is an affiliated program of National ICT Australia.

[‡]Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany. Parts of this paper was written while Swanepoel was visiting the Department of Mechanical Engineering of the University of Melbourne on a Tewkesbury Fellowship.

an edge is its length multiplied by a non-negative *weight*. The weight is determined by a given function of the total flow being routed through that edge, where the function satisfies a number of conditions. The *Gilbert network problem* (GNP) asks for a minimum-cost network spanning a given set of terminals with given flow demands and a given weight function.

A variation on this problem that we will show to be a special case of the GNP occurs when the terminals consist of n sources and a unique sink, and all flows not between a source and the sink are zero. This problem is of intrinsic interest as a natural restriction of the GNP; it is also of interest for its many applications to areas such as drainage networks [7], gas pipelines [1], and underground mining networks [2].

If the weight function is concave and increasing, the resulting minimum network has a tree topology, and provides a directed path from each source to the sink. Such a network can be called an *arborescence*, and we refer to this special case of the GNP as the *Gilbert arborescence problem* (GAP). Traditionally, the term ‘arborescence’ has been used to describe a rooted tree providing directed paths from the unique root (source) to a given set of sinks. Here we are interested in the case where the flow directions are reversed, i.e. flow is from n sources to a unique sink. It is clear, however, that the resulting weights for the two problems are equivalent, hence we will continue to use the term ‘arborescence’ for the latter case. Moreover, if we take the sum of these two cases, and rescale the flows (dividing flows in each direction by 2), then again the weights for the total flow on each edge are the same as in the previous two cases. This justifies our claim that the GAP can be treated as a special case of the GNP. It will be convenient, however, for the remainder of this paper to think of arborescences as networks with a unique sink.

A *minimum Gilbert arborescence* (MGA) is a (global) minimum-cost arborescence for a given set of terminals and flows, and a given cost function. All flows in the network are directed towards the unique sink. In this paper we investigate the local topological structure of Steiner points in MGAs, over a wide range of different metrics and cost-functions. Such an understanding of the topological structure is a key step towards designing efficient algorithms for constructing MGAs.

In Section 2 we specify the nature of the weight function that we consider in this paper, and formally define minimum Gilbert networks and Gilbert arborescences in Minkowski spaces (which generalise Euclidean spaces). In Section 3 we give a general topological characterisation of Steiner points in such networks, for smooth Minkowski spaces. We then apply this characterisation, in Section 4, to the smooth Minkowski plane with a linear weight function to show that in this case all Steiner points have degree 3. In Section 5 we derive a similar result in higher dimensional Euclidean spaces for a wide class of weight functions.

2 Preliminaries

2.1 Minkowski spaces and Steiner trees

The cost functions for the networks we consider in this paper make use of more general norms than simply the Euclidean norm. Hence, we introduce a generalisation of Euclidean spaces, namely finite-dimensional normed spaces or Minkowski spaces. See [12] for an introduction to Minkowski geometry.

A *Minkowski space* (or *finite-dimensional Banach space*) is \mathbb{R}^n endowed with a *norm* $\|\cdot\|$, which is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies

- $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, $\|x\| = 0$ only if $x = 0$,
- $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and
- $\|x + y\| \leq \|x\| + \|y\|$.

We use $\|\cdot\|_2$ to denote the Euclidean (l_2) norm.

We now discuss some aspects of the SMT problem, since this is a special case of the GNP, where all flow are zero. Our terminology for the SMT problem is based on that used in [5]. Let T be a network interconnecting a set $N = \{p_1, \dots, p_n\}$ of points, called *terminals*, in a Minkowski space. A vertex in T which is not a terminal is called a *Steiner point*. Let $G(T)$ denote the *topology* of T , i.e. $G(T)$ represents the graph structure of T but not the embedding of the Steiner points. Then $G(T)$ for a shortest network T is necessarily a tree, since if a cycle exists, the length of T can be reduced by deleting an edge in the cycle. A network with a tree topology is called a *tree*, its links are called *edges*, and its nodes are called *vertices*. An edge connecting two vertices a, b in T is denoted by ab , and its length by $\|a - b\|$.

The *splitting* of a vertex is the operation of disconnecting two edges av, bv from a vertex v and connecting a, b, v to a newly created Steiner point. Furthermore, though the positions of terminals are fixed, Steiner points can be subjected to arbitrarily small movements provided the resulting network is still connected. Such movements are called *perturbations*, and are useful for examining whether the length of a network is minimal.

A *Steiner tree* (ST) is a tree whose length cannot be shortened by a small perturbation of its Steiner points, even when splitting is allowed. By convexity, an ST is a minimum-length tree for its given topology. A *Steiner minimum tree* (SMT) is a shortest tree among all STs. For many Minkowski spaces bounds are known for the maximum possible degree of a Steiner point in an ST, giving useful restrictions on the possible topology of an SMT. For example, in Euclidean space of any dimension every Steiner point in an ST has degree three. Given a set N of terminals, the *Steiner problem* (or *Steiner Minimum Tree problem*) asks for an SMT spanning N .

2.2 Gilbert flows

Gilbert [4] proposed the following generalisation of the Steiner problem in Euclidean space, which we now extend to Minkowski space. Let T be a network interconnecting a set $N = \{p_1, \dots, p_n\}$ of n terminals in a Minkowski

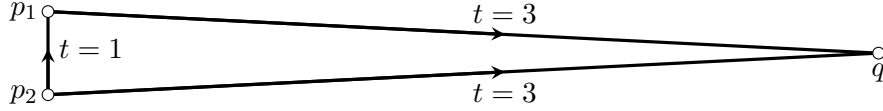


Figure 1: An example where split-routing is cheaper

space. For each pair p_i, p_j , $i \neq j$ of terminals, a non-negative *flow* $t_{ij} = t_{ji}$ is given. The cost of an edge e in T is $w(t_e)l_e$, where l_e is the length of e , t_e is the total flow being routed through e , and $w(\cdot)$ is a unit cost *weight function* defined on $[0, \infty)$ satisfying

$$w(0) \geq 0 \quad \text{and} \quad w(t) > 0 \text{ for all } t > 0, \quad (1)$$

$$w(t_2) \geq w(t_1) \quad \text{for all } t_2 > t_1 \geq 0, \quad (2)$$

$$w(t_1 + t_2) \leq w(t_1) + w(t_2) \quad \text{for all } t_1, t_2 > 0, \quad (3)$$

$$w(\cdot) \text{ is a concave function.} \quad (4)$$

That the function w is concave means by definition that $-w$ is convex. A network satisfying Conditions (1)–(3) is called a *Gilbert network*. For a given edge e in T , $w(t_e)$ is called the *weight* of e , and is also denoted simply by w_e . The *total cost* of a Gilbert network T is the sum of all edge costs, i.e.

$$C(T) = \sum_{e \in E} w(t_e)l_e$$

where E is the set of all edges in T . A Gilbert network T is a *minimum Gilbert network* (MGN), if T has the minimum cost of all Gilbert networks spanning the same point set N , with the same flow demands t_{ij} and the same cost function $w(\cdot)$. By the arguments of [3], an MGN always exists in a Minkowski space when Conditions (1)–(4) are assumed for the weight function.

Conditions (1)–(3) above ensure that the weight function is non-negative, non-decreasing and triangular, respectively. These are natural conditions for most applications. Unfortunately, the first three conditions alone do not guarantee that a minimum Gilbert network is a tree. To show this, we now give an example of a Gilbert network problem with two sources and one sink in the Euclidean plane, where there exists a split-route flow that has a lower cost than any arborescence.

For this example there are two sources p_1, p_2 and a sink q which are the vertices of a triangle $\triangle p_1 p_2 q$ with edge lengths $\|p_1 - p_2\|_2 = 1$ and $\|p_1 - q\|_2 = \|p_2 - q\|_2 = 10$, as illustrated in Figure 1. The tonnages at p_1 and p_2 are 2 and 4, respectively. The weight function is $w(t) = \lceil (3t + 1)/2 \rceil$, i.e., $(3t + 1)/2$ rounded up to the nearest integer. This function is positive, non-decreasing and triangular, but not concave. For the example we only need the following values:

t	1	2	3	4	6
$w(t)$	2	4	5	7	10



Figure 2: The minimum Gilbert arborescence

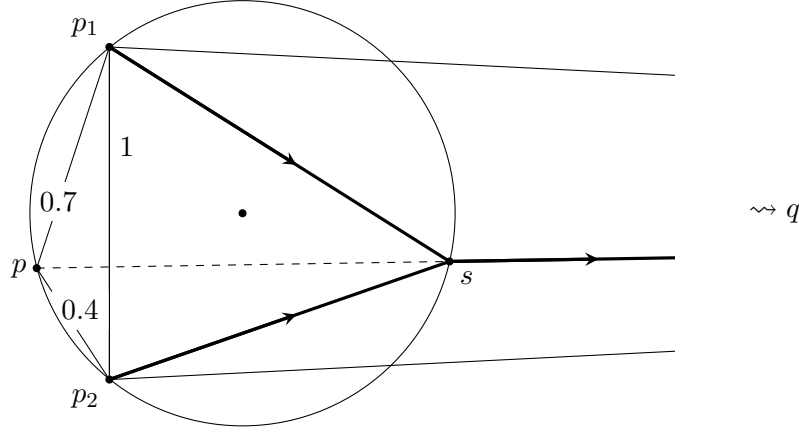


Figure 3: Constructing the weighted Fermat-Torricelli point

Routing 1 unit of the flow from p_2 via p_1 to q gives a Gilbert network (Figure 1) of total cost

$$w(1)\|p_1 - p_2\|_2 + w(3)\|p_1 - q\|_2 + w(3)\|p_2 - q\|_2 = 102.$$

For a Gilbert arborescence we route the tonnages from p_1 and p_2 to q via some point s as in Figure 2. We calculate a minimum Gilbert arborescence by using the weighted Melzak algorithm as described in [4]. To construct the weighted Fermat-Torricelli point s , first construct the unique point p outside $\triangle p_1 p_2 q$ such that $\|p - p_1\|_2 = 0.7$ and $\|p - p_2\|_2 = 0.4$ (Figure 3). Then construct the circumscribed circle of $\triangle p p_1 p_2$, which will intersect the so-called *weighted Simpson line* pq in the required point s . The resulting total cost is

$$\begin{aligned} & w(2)\|p_1 - s\|_2 + w(4)\|p_2 - s\|_2 + w(6)\|s - q\|_2 \\ &= w(6)\|p - q\|_2 = \sqrt{9982.5 + 7\sqrt{3890.25}} \\ &= 102.074\dots, \end{aligned}$$

as some trigonometry will show. This shows that split routing can be necessary when the weight function is not concave.

For the remainder of the paper we assume that the weight function w is concave in addition to the other three conditions. In this case it is known [4, 11] that in the case where there is a single sink there always exists a

minimum Gilbert network that is a Gilbert arborescence. This means that we can (and will) without loss of generality only consider MGAs. (Note that in [3], Condition (3), which we call the *triangular condition*, was incorrectly interpreted as concavity of the cost function.)

The *Gilbert network problem* (GNP) is to find an MGN for a given terminal set N , flows t_{ij} and cost function $w(\cdot)$. Since its introduction in [4], various aspects of the GNP have been studied, although the emphasis has been on discovering geometric properties of MGNs (see [3], [11], [13], [14]). As in the Steiner problem, additional vertices can be added to create a Gilbert network whose cost is less than would otherwise be possible, and these additional points are again called *Steiner points*. A Steiner point s in T is called *locally minimal* if a perturbation of s does not reduce the cost of T . A Gilbert network is called *locally minimal* if no perturbation of the Steiner points reduces the cost of T .

The special case of the Gilbert model that is of interest in this work is when $N = \{p_1, \dots, p_n, q\}$ is a set of terminals in a Minkowski space, where p_1, \dots, p_n are *sources* with respective positive flows t_1, \dots, t_n , and q is the *sink*. All flows are between the sources and the sink; there are no flows between sources. It has been shown in [11] that concavity of the weight function implies that an MGN of this sort is a tree. Hence we refer to an MGN with this flow structure as a *minimum Gilbert arborescence* (MGA), and, as mentioned in the introduction, we refer to the problem of constructing such an MGA as the *Gilbert arborescence problem* (GAP).

If v_1 and v_2 are two adjacent vertices in a Gilbert arborescence, and flow is from v_1 to v_2 then we denote the edge connecting the two vertices by $v_1 v_2$.

3 Characterisation of Steiner Points

In this section, we generalise a theorem of Lawlor and Morgan [6] to give a local characterisation of Steiner points in an MGA. The characterisation in [6] holds for SMTs, which correspond to the case of MGAs with a constant weight function. Their theorem is formulated for arbitrary Minkowski spaces with differentiable norm. Our proof is based on the proof of Lawlor and Morgan's theorem given in [8]. A generalisation to non-smooth norms is contained in [10] for SMTs and in [15] for MGAs. Such a generalisation is much more complicated and involves the use of the subdifferential calculus.

We first introduce some necessary definitions relating to Minkowski geometry, in particular with relation to dual spaces. For more details, see [12].

We denote the inner product of two vectors $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle$. For any given norm $\|\cdot\|$, the dual norm $\|\cdot\|^*$ is defined as follows:

$$\|z\|^* = \sup_{\|x\| \leq 1} \langle z, x \rangle.$$

We say that a Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is *smooth* if the norm is differentiable at any $x \neq o$, i.e., if

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t} =: f_x(h)$$

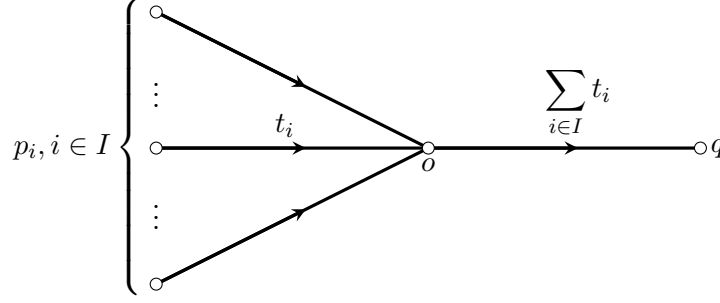


Figure 4: A Gilbert network with star topology.

exists for all $x, h \in \mathbb{R}^n$ with $x \neq o$. It follows easily that f_x is a linear operator $f_x : \mathbb{R}^n \rightarrow \mathbb{R}$ and so can be represented by a vector $x^* \in \mathbb{R}^n$, called the dual vector of x , such that $\langle x^*, y \rangle = f_x(y)$ for all $y \in \mathbb{R}^n$, and $\|x^*\|^* = 1$. In fact x^* is just the gradient of the norm at x , i.e., $x^* = \nabla \|x\|$.

More generally, even if the norm is not differentiable at x , a vector $x^* \in \mathbb{R}^n$ is a *dual vector* of x if x^* satisfies $\langle x^*, x \rangle = \|x\|$ and $\|x^*\|^* = 1$. By the Hahn-Banach separation theorem, each non-zero vector in a Minkowski space has at least one dual vector. A Minkowski space is then smooth if and only if each non-zero vector has a unique dual vector.

A norm is *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ imply that $\|\frac{1}{2}(x + y)\| < 1$, or equivalently, that the *unit sphere*

$$S(\|\cdot\|) = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

does not contain any straight line segment. A norm $\|\cdot\|$ is smooth [strictly convex] if and only if the dual norm $\|\cdot\|^*$ is strictly convex [smooth, respectively].

Theorem 1. *Suppose a smooth Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is given together with a weight function w that satisfies Conditions (1)–(4), sources $p_1, \dots, p_n \in \mathbb{R}^n$, and a single sink $q \in \mathbb{R}^n$, all different from the origin o . Let the flow associated with p_i be t_i . (See Figure 4.) For each p_i let p_i^* denote its dual vector, and let q^* denote the dual vector of q . Then the Gilbert arborescence with edges op_i , $i = 1, \dots, n$ and oq , where all flows are routed via the Steiner point o , is a minimal Gilbert arborescence if and only if*

$$\sum_{i=1}^n w(t_i) p_i^* + w\left(\sum_{i=1}^n t_i\right) q^* = o \quad (5)$$

and

$$\left\| \sum_{i \in I} w(t_i) p_i^* \right\|^* \leq w\left(\sum_{i \in I} t_i\right) \text{ for all } I \subseteq \{1, \dots, n\}. \quad (6)$$

Note: We think of Condition 5 as a flow-balancing condition at the Steiner point, and Condition 6 as a condition that ensures that the Steiner point does not split.

Proof. (\Rightarrow) We are given that the star is not more expensive than any other Gilbert network with the same sources, sink, flows and weight function.

In particular, o is the so-called weighted Fermat-Torricelli point of the $n+1$ points p_1, \dots, p_n, q with weights $t_1, \dots, t_n, \sum_{i=1}^n t_i$, respectively, which implies the balancing condition (5). We include a self-contained proof for completeness. If the Steiner point o is moved to $-te$, where $t \in \mathbb{R}$ and $e \in \mathbb{R}^n$ is a unit vector (in the norm), the resulting arborescence is not better, by the assumption of minimality. Therefore, the function

$$\begin{aligned} \varphi_e(t) &= \sum_{i=1}^n w(t_i)(\|p_i + te\| - \|p_i\|) \\ &\quad + w(\sum_{i=1}^n t_i)(\|q + te\| - \|q\|) \geq 0 \end{aligned}$$

attains its minimum at $t = 0$. For t in a sufficiently small neighbourhood of 0, $p_i + te \neq o$ and $q + te \neq o$, hence φ_e is differentiable. Therefore,

$$\begin{aligned} 0 = \varphi'_e(0) &= \lim_{t \rightarrow 0} \left(\sum_{i=1}^n w(t_i) \frac{\|p_i + te\| - \|p_i\|}{t} \right. \\ &\quad \left. + w(\sum_{i=1}^n t_i) \frac{\|q + te\| - \|q\|}{t} \right) \\ &= \sum_{i=1}^n w(t_i) \langle p_i^*, e \rangle + w(\sum_{i=1}^n t_i) \langle q^*, e \rangle \\ &= \left\langle \sum_{i=1}^n w(t_i) p_i^* + w(\sum_{i=1}^n t_i) q^*, e \right\rangle. \end{aligned}$$

Since this holds for all unit vectors e , (5) follows.

To show (6) for each $I \subseteq \{1, \dots, n\}$, we may assume without loss of generality that $I \neq \emptyset$ and $I \neq \{1, \dots, n\}$. Consider the Gilbert network obtained by splitting the Steiner point into two points o and $+te$ ($t \in \mathbb{R}$, e a unit vector) as follows. Each p_i , $i \notin I$, is still adjacent to o with flow t_i , and q is joined to o with flow $\sum_{i=1}^n t_i$, but now each p_i , $i \in I$, is adjacent to te with flow t_i , and te is adjacent to o with flow $\sum_{i \in I} t_i$, as shown in Figure 5. Since the new network cannot be better than the original star, we obtain that for any unit vector e , the function

$$\psi_e(t) = \sum_{i \in I} w(t_i)(\|p_i - te\| - \|p_i\|) + w(\sum_{i=1}^n t_i)|t| \geq 0$$

attained its minimum at $t = 0$. Although ψ_e is not differentiable at 0, we

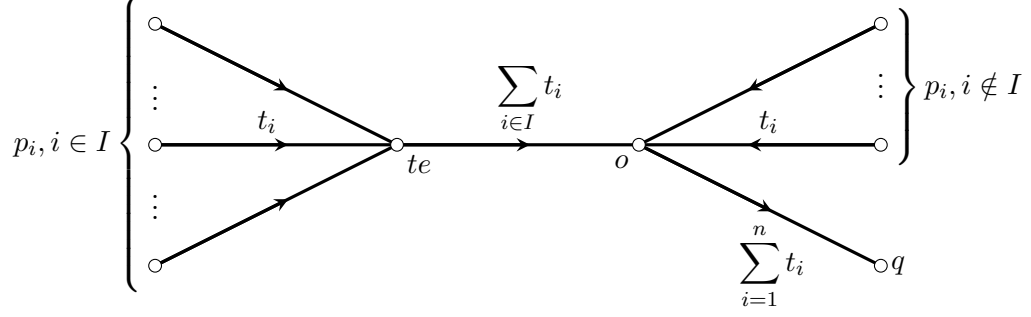


Figure 5: The Gilbert network obtained by splitting the Steiner point o .

can still calculate as follows:

$$\begin{aligned}
0 &\leq \lim_{t \rightarrow 0+} \frac{\psi_e(t)}{t} \\
&= \lim_{t \rightarrow 0+} \sum_{i \in I} w(t_i) \frac{\|p_i - te\| - \|p_i\|}{t} + w\left(\sum_{i=1}^n t_i\right) \\
&= \left\langle \sum_{i \in I} w(t_i) p_i^*, -e \right\rangle + w\left(\sum_{i=1}^n t_i\right).
\end{aligned}$$

Therefore, $\langle \sum_{i \in I} w(t_i) p_i^*, e \rangle \leq w(\sum_{i=1}^n t_i)$ for all unit vectors e , and (6) follows from the definition of the dual norm.

(\Leftarrow) Now assume that p_1^*, p_n^*, q are dual unit vectors that satisfy (5) and (6). Consider an arbitrary Gilbert arborescence T for the given data. For each i , let P_i be the path in T from p_i to q , i.e., $P_i = x_1^{(i)} x_2^{(i)} \dots x_{k_i}^{(i)}$, where $x_1^{(i)} = p_i, x_{k_i}^{(i)} = q$, and $x_j^{(i)} x_{j+1}^{(i)}$ are distinct edges of T for $j = 1, \dots, k_i - 1$. For each edge e of T , let $S_e = \{i : e \text{ is on path } P_i\}$. Then the flow on e is $\sum_{i \in S_e} t_i$ and the total cost of T is

$$\sum_{\substack{e=xy \text{ is} \\ \text{an edge of } T}} w\left(\sum_{i \in S_e} t_i\right) \|x - y\|.$$

The cost of the star is

$$\begin{aligned}
&\sum_{i=1}^n w(t_i) \|p_i\| + w\left(\sum_{i=1}^n t_i\right) \|q\| \\
&= \sum_{i=1}^n w(t_i) \langle p_i^*, p_i \rangle + w\left(\sum_{i=1}^n t_i\right) \langle q^*, q \rangle \\
&= \sum_{i=1}^n w(t_i) \langle p_i^*, p_i - q \rangle \quad \text{by (5)} \\
&= \sum_{i=1}^n w(t_i) \sum_{j=1}^{k_i-1} \langle p_i^*, x_j^{(i)} - x_{j+1}^{(i)} \rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{e=xy \text{ is} \\ \text{an edge of } T}} \left\langle \sum_{i \in S_e} w(t_i) p_i^*, x - y \right\rangle \\
&\leq \sum_{\substack{e=xy \text{ is} \\ \text{an edge of } T}} \left\| \sum_{i \in S_e} w(t_i) p_i^* \right\|^* \|x - y\| \\
&\leq \sum_{\substack{e=xy \text{ is} \\ \text{an edge of } T}} w\left(\sum_{i \in S_e} t_i\right) \|x - y\| \quad \text{by (6)}. \quad \square
\end{aligned}$$

Note that the necessity of the conditions (5) and (6) holds even if the weight function is not concave. It is only in the proof of the sufficiency that we need all minimal Gilbert networks with a single source to be arborescences.

4 Degree of Steiner Points in a Minkowski plane with linear weight function

We now apply the characterisation of the previous section in the two-dimensional case, assuming further that the weight function is linear: $w(t) = d + ht$, $d > 0, h \geq 0$.

Theorem 2. *In a smooth Minkowski plane and assuming a linear weight function $w(t) = d + ht$, $d > 0, h \geq 0$, a Steiner point in an MGA necessarily has degree 3.*

Proof. By Theorem 1, an MGA with a Steiner point of degree $n + 1$ exists in \mathbb{R}^2 with a smooth norm $\|\cdot\|$ if and only if there exist dual unit vectors $p_1^*, \dots, p_n^*, q^* \in \mathbb{R}^2$ such that

$$\sum_{i=1}^n (d + ht_i) p_i^* + (d + h \sum_{i=1}^n t_i) q^* = o$$

and

$$\left\| \sum_{i \in I} (d + ht_i) p_i^* \right\|^* \leq d + h \sum_{i \in I} t_i \quad \text{for all } I \subseteq \{1, \dots, n\}.$$

Label the p_i^* so that they are in order around the dual unit circle. Let $v_i^* = (d + ht_i) p_i^*$ and $w^* = (d + h \sum_{i=1}^n t_i) q^*$. Then the conditions become

$$v_1^* + \dots + v_n^* + w^* = o,$$

and

$$\left\| \sum_{i \in I} v_i^* \right\|^* \leq d + h \sum_{i \in I} t_i \quad \text{for all } I \subseteq \{1, \dots, n\}. \quad (7)$$

Thus we may think of the vectors v_1^*, \dots, v_n^*, w^* as the edges of a convex polygon with vertices $a_j^* = \sum_{i=1}^j v_i^*$, $j = 0, \dots, n$ in this order (see Figure 6).

Assume for the purpose of finding a contradiction that $n > 3$. Then the polygon has at least 4 sides. Note that the diagonals $a_0^* a_j^*$ and $a_{j-1}^* a_n^*$

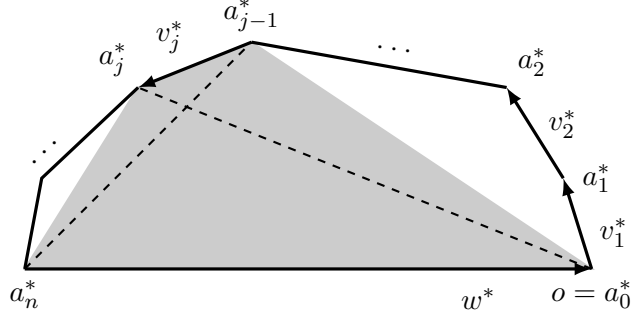


Figure 6: A polygon with edges corresponding to v_j^* .

intersect. Applying the triangle inequality to the two triangles formed by these diagonals and the two edges v_j^* and w^* (as illustrated in Figure 6), we obtain

$$\begin{aligned}
\|a_j^*\|^* + \|a_n^* - a_{j-1}^*\|^* &\geq \|v_j^*\|^* + \|w^*\|^* \\
&= d + ht_j + d + h \sum_{i=1}^n t_i \\
&= d + h \sum_{i=1}^j t_i + d + h \sum_{i=j}^n t_i \\
&\geq \left\| \sum_{i=1}^j v_i^* \right\|^* + \left\| \sum_{i=j}^n v_i^* \right\|^* \quad \text{by (7)} \\
&= \|a_j^*\|^* + \|a_n^* - a_{j-1}^*\|^*.
\end{aligned}$$

Therefore, equality holds throughout, and we obtain equality in the triangle inequality. Since the dual norm is strictly convex, it follows that v_j^* and w^* are parallel. This holds for all $j = 2, \dots, n-1$. It follows that $p_1^* = \dots = p_n^* = -q^*$. Geometrically this means that the unit vectors $\frac{1}{\|p_i\|}p_i$ and $-\frac{1}{\|q\|}q$ all have the same supporting line on the unit ball. We can think of this condition on the vectors p_i and q as a generalisation of collinearity to Minkowski space.

Choose a point s_2 on the edge op_i such that the line through s_2 parallel to op_n intersects the edge op_1 in s_1 , say, with $s_1 \neq o$. See Figure 7. Because of the straight line segments on the boundary of the unit ball, $\|x + y\| = \|x\| + \|y\|$ for any x, y such that the unit vectors $\frac{1}{\|x\|}x$ and $\frac{1}{\|y\|}y$ lie on this segment. In particular,

$$\|s_2 - s_1\| + \|s_1 - o\| = \|s_2 - o\|. \quad (8)$$

Now replace p_2o by the edges p_2s_2 and s_2s_1 , replace p_1o by p_1s_1 and s_1o , and add the flow t_2 to s_1o . The change in cost in the new Gilbert arborescence

such that

$$\|v_i\|_2 = w(t_i), \quad i = 1, \dots, m \quad (9)$$

$$\left\| \sum_{i=1}^m v_i \right\|_2 = w\left(\sum_{i=1}^m t_i\right), \quad (10)$$

$$\forall I \subseteq \{1, \dots, m\} \text{ with } 2 \leq |I| \leq m-2, \quad \left\| \sum_{i \in I} v_i \right\|_2 \leq w\left(\sum_{i \in I} t_i\right). \quad (11)$$

Theorem 4. *If the weight function $w(\cdot)$ satisfies Conditions (1)–(4), and is differentiable with $(w^2)'$ convex and $w(0) > 0$, then all Steiner points in MGAs have degree 3.*

Note that the hypothesis is indeed satisfied for the weight function $w(t) = d + ht^\alpha$ for any $d, h > 0$ and $\alpha \in [0, 1/2] \cup \{1\}$, but not when $\alpha \in (1/2, 1)$.

For the proof of Theorem 4 we need the following inequality valid for functions with convex derivative.

Lemma 5. *If $f : [0, \infty) \rightarrow \mathbb{R}$ is differentiable with f' convex, then for all $m \geq 2$ and all $t_i \geq 0$ ($i = 1, 2, \dots, m$),*

$$\begin{aligned} \frac{(m-1)(m-2)}{2} f(0) + \sum_{1 \leq i < j \leq m} f(t_i + t_j) \\ \leq (m-2) \sum_{i=1}^m f(t_i) + f\left(\sum_{i=1}^m t_i\right). \end{aligned} \quad (12)$$

Note that in (12), as well as in the sequel, the summation $\sum_{1 \leq i < j \leq m}$ means that the sum is over all $m(m-1)/2$ pairs (i, j) that satisfy $1 \leq i < j \leq n$. We postpone the proof of the lemma.

Proof of Theorem 4. Suppose that a Steiner point of degree $m+1 \geq 3$ exists. We intend to show that $m+1 = 3$. Note that we do not only consider the case $m+1 = 4$, since it is *a priori* possible that degree 4 Steiner points don't exist, although degree 5 points exist (although we don't have any examples, and it seems highly unlikely).

By Corollary 3 there exist vectors v_1, \dots, v_m and numbers $t_1, \dots, t_m > 0$ that satisfy (9), (10) and (11). Square (10):

$$\begin{aligned} \left(w\left(\sum_{i=1}^m t_i\right) \right)^2 &= \left\| \sum_{i=1}^m v_i \right\|_2^2 \\ &= \sum_{i=1}^m \|v_i\|_2^2 + 2 \sum_{1 \leq i < j \leq m} \langle v_i, v_j \rangle \\ &= \sum_{i=1}^m (w(t_i))^2 + 2 \sum_{1 \leq i < j \leq m} \langle v_i, v_j \rangle \quad (\text{by (9)}). \end{aligned} \quad (13)$$

Estimate $\langle v_i, v_j \rangle$ by applying (11) to $I = \{i, j\}$:

$$\begin{aligned} 2 \langle v_i, v_j \rangle &= \|v_i + v_j\|_2^2 - \|v_i\|_2^2 - \|v_j\|_2^2 \\ &\leq w(t_i + t_j)^2 - w(t_i)^2 - w(t_j)^2 \quad (\text{again by (9)}). \end{aligned}$$

Sum this inequality over all pairs (i, j) with $1 \leq i < j \leq m$:

$$\begin{aligned} 2 \sum_{1 \leq i < j \leq m} \langle v_i, v_j \rangle &\leq \sum_{1 \leq i < j \leq m} (w(t_i + t_j)^2 - w(t_i)^2 - w(t_j)^2) \\ &= \sum_{1 \leq i < j \leq m} (w(t_i + t_j))^2 - (m-1) \sum_{i=1}^m (w(t_i))^2, \end{aligned}$$

since each $(w(t_i))^2$ is summed once for each of the $m-1$ pairs in which i appears. Substitute this into (13):

$$\left(w \left(\sum_{i=1}^m t_i \right) \right)^2 \leq \sum_{1 \leq i < j \leq m} (w(t_i + t_j))^2 - (m-2) \sum_{i=1}^m (w(t_i))^2. \quad (14)$$

Apply Lemma 5 to $f = w^2$:

$$\begin{aligned} \frac{(m-1)(m-2)}{2} (w(0))^2 + \sum_{1 \leq i < j \leq m} (w(t_i + t_j))^2 \\ \leq (m-2) \sum_{i=1}^m (w(t_i))^2 + \left(w \left(\sum_{i=1}^m t_i \right) \right)^2. \end{aligned} \quad (15)$$

Combining (14) and (15), we obtain $\frac{(m-1)(m-2)}{2} (w(0))^2 \leq 0$, which implies $m+1 \leq 3$. \square

Proof of Lemma 5. We use induction on $m \geq 2$. The base case $m = 2$ is trivial.

Assume now that $m \geq 3$ and that the lemma holds for $m-1$, in particular that

$$\begin{aligned} \frac{(m-2)(m-3)}{2} f(0) + \sum_{1 \leq i < j \leq m-1} f(t_i + t_j) \\ \leq (m-3) \sum_{i=1}^{m-1} f(t_i) + f \left(\sum_{i=1}^{m-1} t_i \right). \end{aligned} \quad (16)$$

Consider $x := t_m$ to be variable and t_1, \dots, t_{m-1} fixed. Set $T := \sum_{i=1}^{m-1} t_i$, and define

$$\begin{aligned} g(x) &:= (m-2) \sum_{i=1}^m f(t_i) + f \left(\sum_{i=1}^m t_i \right) - \sum_{1 \leq i < j \leq m} f(t_i + t_j) \\ &= (m-2) \sum_{i=1}^{m-1} f(t_i) + (m-2) f(x) + f(T+x) \\ &\quad - \sum_{1 \leq i < j \leq m-1} f(t_i + t_j) - \sum_{i=1}^{m-1} f(t_i + x). \end{aligned}$$

We have to show that $g(x) \geq \frac{(m-1)(m-2)}{2}f(0)$ for all $x > 0$. First of all,

$$\begin{aligned}
g(0) &= (m-2) \sum_{i=1}^{m-1} f(t_i) + (m-2)f(0) + f(T) \\
&\quad - \sum_{1 \leq i < j \leq m-1} f(t_i + t_j) - \sum_{i=1}^{m-1} f(t_i) \\
&= (m-3) \sum_{i=1}^{m-1} f(t_i) + f(T) - \sum_{1 \leq i < j \leq m-1} f(t_i + t_j) + (m-2)f(0) \\
&\geq \frac{(m-2)(m-3)}{2}f(0) + (m-2)f(0) \quad (\text{by (16)}) \\
&= \frac{(m-1)(m-2)}{2}f(0).
\end{aligned}$$

It is therefore sufficient to show that $g'(x) \geq 0$ for all $x > 0$. We have

$$g'(x) = (m-2)f'(x) + f'(T+x) - \sum_{i=1}^{m-1} f'(t_i + x).$$

If $T = 0$ then $t_i = 0$ for all $i = 1, \dots, m-1$, and then $g'(x)$ is identically 0. We may therefore assume without loss of generality that $T > 0$. Write each $x + t_j$ as a convex combination of x and $x + T$:

$$t_j + x = \left(1 - \frac{t_j}{T}\right)x + \frac{t_j}{T}(x + T).$$

Since f' is convex,

$$\begin{aligned}
\sum_{j=1}^{m-1} f'(t_j + x) &\leq \sum_{j=1}^{m-1} \left(1 - \frac{t_j}{T}\right) f'(x) + \frac{t_j}{T} f'(x + T) \\
&= (m-2)f'(x) + f'(x + T),
\end{aligned}$$

which gives $g'(x) \geq 0$. This finishes the induction step and the proof. \square

We now show that for each $\alpha \in (1/2, 1)$ there exist Gilbert arborescences with Steiner points of degree 4 if the weight function is $w(t) = d + ht^\alpha$, with $d, h > 0$ chosen appropriately. In the example all incoming tonnages will be equal. We first use Corollary 3 to formulate a result for general weight functions.

Proposition 6. *Let $w(\cdot)$ be a weight function that satisfies Conditions (1)–(4). There exists an MGA with degree 4 in Euclidean 3-space with equal tonnages $t_1 = t_2 = t_3 =: t$ and with weight function $w(\cdot)$ if, and only if*

$$3w(t)^2 + w(3t)^2 \leq 3w(2t)^2. \quad (17)$$

Proof. By Corollary 3, an MGA of degree 4 exists in Euclidean space with equal tonnages $t_1 = t_2 = t_3 =: t$ if, and only if, there exist three Euclidean vectors v_1, v_2, v_3 such that

$$\|v_1\|_2 = \|v_2\|_2 = \|v_3\|_2 = w(t), \quad (18)$$

$$\|v_1 + v_2\|_2, \|v_2 + v_3\|_2, \|v_1 + v_3\|_2 \leq w(2t), \quad (19)$$

$$\|v_1 + v_2 + v_3\|_2 = w(3t). \quad (20)$$

(These vectors will of course span a space of dimension at most 3.)

Square (20) and use (18) to obtain an expression for the sum of the three inner products:

$$2(\langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \langle v_1, v_3 \rangle) = w(3t)^2 - 3w(t)^2. \quad (21)$$

Square (19) and use (18) to obtain an upper bound on each inner product $\langle v_i, v_j \rangle$ (geometrically this is a lower bound on the angle between any two vectors):

$$2 \langle v_i, v_j \rangle \leq w(2t)^2 - 2w(t)^2,$$

and substitute this into (21) to obtain (17).

Conversely, if we assume (17), we have to find three vectors that satisfy (18)–(20). Note that for each $\lambda \in [-1/2, 1]$ there exist three unit vectors $u_1, u_2, u_3 \in \mathbb{R}^3$ such that the inner product of each pair equals λ . The one extreme $\lambda = 1$ corresponds to three equal vectors, and the other extreme $\lambda = -1/2$ to three coplanar vectors such that any two are at an angle of 120° . If we set

$$\lambda := \frac{w(3t)^2}{6w(t)^2} - \frac{1}{2},$$

then the vectors $v_i := w(t)u_i$ satisfy (18)–(20). It remains to show that this value of λ really lies in the interval $[-1/2, 1]$. The lower bound $\lambda \geq -1/2$ holds trivially, while the upper bound $\lambda \leq 1$ follows from $0 \leq w(3t) \leq 3w(t)$, which in turn follows from non-negativity (Condition (1)) and concavity (Condition (4)) of the cost function. \square

Corollary 7. *For each $\alpha \in (1/2, 1)$ there exists $d, h, t > 0$ and an MGA of degree 4 in Euclidean 3-space with cost function $w(t) = d + ht^\alpha$, and tonnages $t_1 = t_2 = t_3 := t$.*

Proof. By choosing the unit of the weight function appropriately, we may assume without loss of generality that $w(t) = D + t^\alpha$. We may similarly assume that $t = 1$, and then by Proposition 6, we only have to show that $3w(1)^2 + w(3)^2 \leq 3w(2)^2$ will hold for some value of $D > 0$, which is

$$3(D + 1)^2 + (D + 3^\alpha)^2 \leq 3(D + 2^\alpha)^2,$$

or equivalently,

$$D^2 + (6 + 2 \cdot 3^\alpha - 6 \cdot 2^\alpha)D + 3 + 3^{2\alpha} - 3 \cdot 2^{2\alpha} \leq 0.$$

A sufficient condition for this to hold for some $D > 0$, is that the quadratic polynomial in D on the left has a positive root. For this to hold it is in turn sufficient that its constant coefficient is negative, i.e., that

$$f(\alpha) = 3 + 3^{2\alpha} - 3 \cdot 2^{2\alpha} < 0 \text{ for all } \alpha \in (1/2, 1).$$

However, it is easily checked that $f(1/2) = f(1) = 0$ and that $f''(\alpha) > 0$ for all $\alpha \in (1/2, 1)$, so that f is convex on $(1/2, 1)$. It follows that f is negative on $(1/2, 1)$, which finishes the proof. \square

6 Conclusion

In this paper we have studied the problem of designing a minimum cost flow network interconnecting n sources and a single sink, each with known locations and flows, in general finite-dimensional normed spaces. The network may contain other unprescribed nodes, known as Steiner points. For concave increasing cost functions, a minimum cost network of this sort has a tree topology, and hence can be called a Minimum Gilbert Arborescence (MGA). We have characterised the local topological structure of Steiner points in MGAs for linear weight functions, specifically showing that Steiner points necessarily have degree 3.

References

- [1] S. Bhaskaran, and F. J. M. Salzborn, “Optimal design of gas pipeline networks”, *Journal of the Operational Research Society*, vol. 30, pp. 1047-1060, 1979.
- [2] M. Brazil, D. H. Lee, J. H. Rubinstein, D. A. Thomas, J. F. Weng, and N. C. Wormald, “Network optimisation of underground mine design”, *The Australian Institute of Mining and Metallurgy Proceedings*, vol. 305, pp. 57-65, 2000.
- [3] C. L. Cox, “Flow-dependent networks: Existence and behavior at Steiner points”, *Networks*, vol. 31, pp. 149-156, 1998.
- [4] E. N. Gilbert, “Minimum cost communication networks”, *Bell System Technical Journal*, vol. 46, pp. 2209-2227, 1967.
- [5] F. K. Hwang, D. S. Richards, and P. Winter, “The Steiner tree problem”, *Annals of Discrete Mathematics vol. 53*, Elsevier Science Publishers, 1992.
- [6] G. R. Lawlor, and F. Morgan, “Paired calibrations applied to soap films, immiscible fluids, and surfaces and networks minimizing other norms”, *Pacific Journal of Mathematics*, vol. 166, pp. 55-82, 1994.
- [7] D. H. Lee, “Low cost drainage networks”, *Networks*, vol. 6, pp. 351-371, 1976.

- [8] K. J. Swanepoel, “Vertex degrees of Steiner Minimal Trees in ℓ_p^d and other smooth Minkowski spaces”, *Discrete Comput. Geom.*, vol. 21, 437-447, 1999.
- [9] K. J. Swanepoel, “The local Steiner problem in normed planes”, *Networks*, vol. 36, pp. 104-113, 2000.
- [10] K. J. Swanepoel, “The local Steiner problem in finite-dimensional normed spaces”, *Discrete & Computational Geometry*, vol. 37 pp. 419-442, 2007.
- [11] D. A. Thomas, and J. F. Weng, “Minimum cost flow-dependent communication networks”, *Networks*, vol. 48, pp. 39-46, 2006.
- [12] A. C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and its Applications 63, Cambridge University Press, 1996.
- [13] D. Trietsch, “Minimal Euclidean networks with flow dependent costs - the generalized Steiner case”, Discussion Paper No. 655, *Center for Mathematical Studies in Economics and Management Science*, Northwestern University, Evanston, Illinois, 1985.
- [14] D. Trietsch, and J. F. Weng, “Pseudo-Gilbert-Steiner trees”, *Networks*, vol. 33, pp. 175-178, 1999.
- [15] M. Volz, “Gradient-constrained flow-dependent networks for underground mine design”, PhD thesis, Department of Electrical and Electronic Engineering, University of Melbourne, 2008.
- [16] A. Weber, “Über den Standort der Industrien”, Tübingen, Germany, 1909. (translated by C. J. Friedrich as *Alfred Weber's Theory of the Location of Industries*, Chicago University Press, Chicago, 1929.)